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Clebsch–Gordan problem for the three-dimensional Lorentz group in the elliptic basis: II. Tensor products involving discrete series

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Abstract. Following the approach of Kerimov and Verdiyev (1998 *J. Phys. A: Math. Gen.* **31** 3573) the Clebsch–Gordan (CG) coefficients in a $SO(2)$ basis for tensor products involving discrete series representations of $SO_0(2, 1)$ are calculated. A completeness relation for CG coefficients for all the cases under consideration is derived.

1. Introduction

In a previous paper [1], referred to hereafter as I, we suggested a new approach to the Clebsch–Gordan (CG) problem for the unitary representations of the semi-simple Lie group. In this approach, the CG coefficients are defined as the matrix element of an intertwining operator between a tensor product representation and an irreducible component occurring in the reduction. In that paper, we discussed the problem of decomposing the tensor product of two irreducible representations of $G = SO_0(2, 1)$ in a $SO(2)$ basis in the cases where the two representations both belong to the principal series, or if both belong to the complementary series, or if one of them belongs to each of these classes. The object of this paper is to extend this study to all of the remaining cases of the tensor product. (The notation of I will be followed here.)

In this series of papers, we show that the CG coefficients for all cases of the tensor product of unitary irreducible representations (UIRs) of G in a $SO(2)$ basis are expressible in terms of a single function; namely in terms of the bilateral series ${}_3H_3(1)$ with the unit argument defined in the complex space C_3 of the variable j_1, j_2, j. This case is due to the facts that [2] (a) the intertwining operator, which displays the decomposition of the tensor product of two principal series representations, also exists for the tensor product of elementary representations (but it need no longer be a unitary operator); (b) the complementary series (discrete series) occur as unitarizations of elementary representations (as unitarizations of the quotient of such representations). The function ${}_3H_3(1)$ is singular on a subset of discrete points of C_3, corresponding to the cases $T^+_{l_1} \otimes T^+_{l_2}$ and $T^-_{l_1} \otimes T^-_{l_2}$ (see, section 4). In general, the ${}_3H_3(1)$ function can be expressed in terms of two generalized hypergeometric functions ${}_3F_2$ with unit argument; however, it reduces to the single ${}_3F_2(1)$ function when at least one of the coupling UIRs belongs to a discrete series.

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The contents of this paper are arranged as follows. In section 2 the decomposition of the tensor product of a discrete series representation with a representation of principal series is described. In section 3 we discuss the tensor product of complementary series representation with a representation of discrete series. The tensor products of discrete series representations are studied in section 4.

2. Tensor products of discrete series and principal series representations

In this section we shall discuss the tensor products $T^{\pm}_{l_1} \otimes T_{\mathrm{i}\rho_2-1/2}$ of the discrete series representations $T^{\pm}_{l_1}$, $l_1 = 0, 1, 2, \ldots$ with representation of the principal series $T_{\mathrm{i}\rho_2-1/2}$, $\rho_2 \geqslant 0$.

Let $C^{\infty}(S^1 \times S^1)$ be a space of infinitely differentiable functions $f(s,t)$ on $S^1 \times S^1$. Define an operator A_{l_1}, $l_1 = 0, 1, 2, \ldots$ on $C^{\infty}(S^1 \times S^1)$ by

$$(A_{l_1} f)(s,t) = \frac{2^{l_1}}{\sqrt{\pi}} \lim_{j_1 \to l_1} \frac{\Gamma(j_1+1)}{\Gamma\left(-\frac{2j_1+1}{2}\right)} \int_{S^1} |[s_1, s]|^{-j_1-1} f(s_1, t)\, \mathrm{d}s_1 \tag{2.1}$$

and let H_0 be a quotient space of $C^{\infty}(S^1 \times S^1)$ with respect to the kernel of A_{l_1} (i.e. $H_0 = C^{\infty}(S^1 \times S^1)/\ker A_{l_1}$). Let $H_{l_1,\mathrm{i}\rho_2-1/2}$ be Hilbert space completion of H_0 with respect to the scalar product induced by the sesquilinear functional

$$(f_1, f_2) = \frac{2^{l_1}}{\sqrt{\pi}} \lim_{j_1 \to l_1} \frac{\Gamma(j_1+1)}{\Gamma\left(-\frac{2j_1+1}{2}\right)} \int_{S^1 \times S^1 \times S^1} |[s_1, s]|^{-j_1-1} f_1(s_1, t) \overline{f_2(s_2, t)}\, \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}t \tag{2.2}$$

where $f_1, f_2 \in C^{\infty}(S^1 \times S^1)$. (Actually, the space $H_{l_1,\mathrm{i}\rho_2-1/2}$ does not change with ρ_2 but we retain these labels to remind us of the representation being constructed.) Then the space $H_{l_1,\mathrm{i}\rho_2-1/2}$ can be made to carry a representation $V_{l_1,\mathrm{i}\rho_2-1/2} \equiv (T^{+}_{l_1} \oplus T^{-}_{l_1}) \otimes T_{\mathrm{i}\rho_2-1/2}$ ($l_1 \geqslant 0$ integer) of the G in the following way. Let $T_{j_1} \otimes T_{j_2}$ ($j_1, j_2 \in C$) be the representation of G on $C^{\infty}(S^1 \times S^1)$, with G-action defined in (I.3.3):

$$(T_{j_1} \otimes T_{j_2}(g) f)(s,t) = (sg)_3^{j_1} (tg)_3^{j_2} f(s_g, t_g) \qquad g \in G \tag{2.3}$$

where $s_g = (sg)/(sg)_3$ and $t_g = (tg)/(tg)_3$. Then the representation $V_{l_1,\mathrm{i}\rho_2-1/2}$ occurs as a unitarization of a representation on H_0 defined by $T_{j_1} \otimes T_{j_2}$, $j_1 = l_1$, $j_2 = \mathrm{i}\rho_2 - \frac{1}{2}$. Now let H_0^{+} and H_0^{-} be the subspaces of H_0 consisting of the elements whose representative is the functions $f \in C^{\infty}(S^1 \times S^1)$ of the form

$$f(s,t) = \sum_{m_1=l+1}^{\infty} \sum_{m_2=-\infty}^{\infty} a_{m_1 m_2} \exp(\mathrm{i}m_1\varphi_1 + \mathrm{i}m_2\varphi_2) \tag{2.4}$$

and

$$f(s,t) = \sum_{m_1=-\infty}^{-l-1} \sum_{m_2=-\infty}^{\infty} a_{m_1 m_2} \exp(\mathrm{i}m_1\varphi_1 + \mathrm{i}m_2\varphi_2) \tag{2.5}$$

respectively, where

$$s = (\cos\varphi_1, \sin\varphi_1, 1) \qquad t = (\cos\varphi_2, \sin\varphi_2, 1). \tag{2.6}$$

Let $H^{\pm}_{l_1,\mathrm{i}\rho_2-1/2}$ be the subspace of $H_{l_1,\mathrm{i}\rho_2-1/2}$ generated by $H_0^{\pm}$. The subspaces $H^{\pm}_{l_1,\mathrm{i}\rho_2-1/2}$ are invariant under $V_{l_1,\mathrm{i}\rho_2-1/2}$. Denote by $V^{\pm}_{l_1,\mathrm{i}\rho_2-1/2}$ the subrepresentations of $V_{l_1,\mathrm{i}\rho_2-1/2}$ defined on $H^{\pm}_{l_1,\mathrm{i}\rho_2-1/2}$. Then the representations $V^{\pm}_{l_1,\mathrm{i}\rho_2-1/2}$ are unitarily equivalent to $T^{\pm}_{l_1} \otimes T_{\mathrm{i}\rho_2-1/2}$.

Now consider the sesquilinear functional $(\Phi_1, \Phi_2)_{l_1,\mathrm{i}\rho_2-1/2}$ on $K(X)$

$$(\Phi_1, \Phi_2) = \int_X \Phi_1(x) B_{l_1,\mathrm{i}\rho_2-1/2}(U_{\mathrm{i}\rho_2-l_1-1/2}(\alpha_x)\Phi_2)\,\mathrm{d}x \tag{2.7}$$

where $\Phi_1, \Phi_2 \in K(X)$ and $B_{l_1,\mathrm{i}\rho_2-1/2}$ is the generalized function on X:

$$B_{l_1,\mathrm{i}\rho_2-1/2}(\Phi) = \frac{2^{l_1}}{\sqrt{\pi}} \lim_{j_1\to l_1} \frac{\Gamma(j_1+1)}{\Gamma\left(-\frac{2j_1+1}{2}\right)} \int_{-\infty}^{\infty} (q^2+1)^{\frac{2j_1+2\mathrm{i}\rho_2+1}{4}} |q|^{-2j_1-2} \Phi(1,q,q)\,\mathrm{d}q \tag{2.8}$$

where $\Phi \in K(X)$. Let F_0 be a quotient space of $K(X)$ with respect to the kernel of the sesquilinear functional (2.7). Let $F_{l_1,\mathrm{i}\rho_2-1/2}$ be the Hilbert space completion of F_0 with respect to the scalar product induced by the sesquilinear functional (2.7). Then the representation $V_{l_1,\mathrm{i}\rho_2-1/2}$ is unitarily equivalent to a unitary representation of G acting in the space $F_{l_1,\mathrm{i}\rho_2-1/2}$, which is obtained by extension of $U_{\mathrm{i}\rho_2-l_1-1/2}$. (In what follows, for the sake of simplicity, we shall denote the corresponding extension of a representation by the same symbol.) The mapping exhibiting this equivalence is induced by $P_{l_1,\mathrm{i}\rho_2-1/2}$ (see equation (I.3.5)). Let $F^{\pm}_{l_1,\mathrm{i}\rho_2-1/2}$ be the image of $H^{\pm}_{l_1,\mathrm{i}\rho_2-1/2}$ under the mapping and let $U^{\pm}_{\mathrm{i}\rho_2-l_1-1/2}$ be subrepresentations of $U_{\mathrm{i}\rho_2-l_1-1/2}$ defined on $H^{\pm}_{l_1,\mathrm{i}\rho_2-1/2}$. Then the representations $V^{\pm}_{l_1,\mathrm{i}\rho_2-1/2}$ and $U^{\pm}_{\mathrm{i}\rho_2-l_1-1/2}$ are unitarily equivalent. According to this, the problem of decomposing the tensor products $T^{\pm}_{l_1} \otimes T_{\mathrm{i}\rho_2-1/2}$ is reduced to that of decomposing $U^{\pm}_{\mathrm{i}\rho_2-l_1-1/2}$. Moreover, the intertwining mapping given by (I.3.7) induced a unitary operator from $F^{\pm}_{l_1,\mathrm{i}\rho_2-1/2}$ into a direct integral of the carrier spaces in which the irreducible representations are realized (for details see [2, 3]).

For $f \in H^{\pm}_{l_1,\mathrm{i}\rho_2-1/2}$, the following linear combination of the Fourier components of the element f

$$\tfrac{1}{2} \sum_{\varepsilon=0,1/2} [1 + (-1)^{l_1+2\varepsilon} \mathrm{e}^{\mp\mathrm{i}(j_2+j)}] \gamma^{-1}(j_2 - l_1, j, \varepsilon) C^{j\varepsilon}_{l_1 j_2} f \tag{2.9}$$

is zero (i.e. $C^{j\varepsilon}_{l_1 j_2}$, $\varepsilon = 0, \frac{1}{2}$ are linearly dependent), where $j_2 = \mathrm{i}\rho_2 - \frac{1}{2}$, $j = \mathrm{i}\rho - \frac{1}{2}$ and $C^{j\varepsilon}_{j_1 j_2} f$ are defined in (I.3.10). Consequently, every one of the principal series representation $T_{\mathrm{i}\rho-1/2}$ appears in the tensor product $T^{\pm}_{l_1} \otimes T_{\mathrm{i}\rho_2-1/2}$ once. On the other hand the expression (2.9) at $j = l$, $l = 0, 1, 2, \ldots$ is nothing but $C^{\mp l}_{l_1,\mathrm{i}\rho_2-1/2}$ (see equation (I.3.15)). Therefore, there is no negative discrete series T^{-}_{l} in the decomposition of the tensor product $T^{+}_{l_1} \otimes T_{\mathrm{i}\rho_2-1/2}$ and no positive discrete series T^{+}_{l} for $T^{-}_{l_1} \otimes T_{\mathrm{i}\rho_2-1/2}$. Thus, the structure of the CG series for $T^{\pm}_{l_1} \otimes T_{\mathrm{i}\rho_2-1/2}$ has the form

$$T^{\pm}_{l_1} \otimes T_{\mathrm{i}\rho_2-1/2} = \int_0^{\infty} T_{\mathrm{i}\rho_2-1/2}\,\mathrm{d}\rho \otimes \sum_{l=0}^{\infty} T^{\pm}_{l}. \tag{2.10}$$

There are two types of CG coefficients to be calculated for each case; namely $C^{\mathrm{i}\rho-1/2}_{l_1+,\mathrm{i}\rho_2-1/2}(m; m_1, m_2)$, $C^{l+}_{l_1+,\mathrm{i}\rho_2-1/2}(m; m_1, m_2)$ for $T^{+}_{l_1} \otimes T_{\mathrm{i}\rho_2-1/2}$ and $C^{\mathrm{i}\rho-1/2,\varepsilon}_{l_1-,\mathrm{i}\rho_2-1/2}(m; m_1, m_2)$, $C^{l-}_{l_1-,\mathrm{i}\rho_2-1/2}(m; m_1, m_2)$ for $T^{-}_{l_1} \otimes T_{\mathrm{i}\rho_2-1/2}$, where $\varepsilon = 0, \frac{1}{2}$ enumerates two (linearly dependent) CG coefficients, corresponding to a representation of the principal series. Following the approach of a previous paper we obtain the integral representations for these CG coefficients; the steps are identical to those leading to equation (3.23) of paper I, and the results are

$$C^{\mathrm{i}\rho-1/2,\varepsilon}_{l_1\pm,\mathrm{i}\rho_2-1/2}(m; m_1, m_2) = \frac{1}{(2\pi)^{3/2}} \left[\frac{\Gamma(-l_1 \pm m_1)\Gamma(1/2 - \mathrm{i}\rho_2 + m_2)\Gamma(1/2 + \mathrm{i}\rho + m)}{\Gamma(1 + l_1 \pm m_1)\Gamma(1/2 + \mathrm{i}\rho_2 + m_2)\Gamma(1/2 - \mathrm{i}\rho + m)}\right]^{1/2}$$
$$\times \int_{S^1\times S^1\times S^1} K_\varepsilon(l_1 s, \mathrm{i}\rho_2 - \tfrac{1}{2}, t; \mathrm{i}\rho - \tfrac{1}{2}, u)$$

$$\times \exp(\mathrm{i}m_1\varphi_1 + \mathrm{i}m_2\varphi_2 - \mathrm{i}m\varphi)\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\mathrm{d}\varphi \tag{2.11}$$

$$C^{l\pm}_{l_1\pm,\mathrm{i}\rho_2-1/2}(m; m_1, m_2) = \frac{1}{(2\pi)^{3/2}}\left[\frac{\Gamma(-l_1 \pm m_1)\Gamma(1/2 - \mathrm{i}\rho_2 + m_2)\Gamma(1 + l \pm m)}{\Gamma(1 + l_1 \pm m_1)\Gamma(1/2 + \mathrm{i}\rho_2 + m_2)\Gamma(-l \pm m)}\right]^{1/2} \times \int_{S^1\times S^1\times S^1} K^{\pm}(l_1 s, \mathrm{i}\rho_2 - \tfrac{1}{2}, t; lu)\exp(\mathrm{i}m_1\varphi_1 + \mathrm{i}m_2\varphi_2 - \mathrm{i}m\varphi)\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\mathrm{d}\varphi \tag{2.12}$$

where parameter ε in (2.11) may be any number from $\{0, \frac{1}{2}\}$; $K_\varepsilon(j_1 s, j_2 t; ju)$ and $K^{\pm}(j_1 s, j_2 t; lu)$ are defined by equations (I.3.11) and (I.3.16):

$$K_\varepsilon(j_1 s, j_2 t; ju) = 2^{-1-\frac{j_1+j_2}{2}+j}\left|\sin\frac{\varphi_1 - \varphi}{2}\right|^{-2a_1}\mathrm{sign}^{2\varepsilon}\sin\frac{\varphi_1 - \varphi}{2}\left|\sin\frac{\varphi - \varphi_2}{2}\right|^{-2a_2} \times \mathrm{sign}^{2\varepsilon}\sin\frac{\varphi - \varphi_2}{2}\left|\sin\frac{\varphi_1 - \varphi_2}{2}\right|^{-2a}\mathrm{sign}^{2\varepsilon}\sin\frac{\varphi_1 - \varphi_2}{2} \tag{2.13}$$

$$K^{\pm}(j_1 s, j_2 t; lu) = \frac{1}{2}\sum_{\varepsilon=0}^{1/2}[1 + (-1)^{l+2\varepsilon}\mathrm{e}^{\pm \mathrm{i}\pi z}]K_\varepsilon(j_1 s, j_2 t; lu) \tag{2.14}$$

where $s = (\cos\varphi_1, \sin\varphi_1, 1)$, $t = (\cos\varphi_2, \sin\varphi_2, 1)$, $u = (\cos\varphi, \sin\varphi, 1)$ and

$$-2a = -2 - j_1 - j_2 - j \qquad -2a_1 = z + j \qquad -2a_2 = -z + j \qquad z = j_2 - j_1. \tag{2.15}$$

Furthermore, we have the following completeness relation for CG coefficients:

$$\sum_{m=-\infty}^{\infty}\int_0^{\infty}\omega^{+}(\mathrm{i}\rho - \tfrac{1}{2}, \varepsilon)C^{\mathrm{i}\rho-1/2,\varepsilon}_{l_1+,\mathrm{i}\rho_2-1/2}(m; m_1, m_2)\overline{C^{\mathrm{i}\rho-1/2,\varepsilon}_{l_1+,\mathrm{i}\rho_2-1/2}(m; m_1', m_2')}\,\mathrm{d}\rho + \sum_{l=0}^{\infty}\omega_l^{+}\sum_{m=l+1}^{\infty}C^{l+}_{l_1+,\mathrm{i}\rho_2-1/2}(m; m_1, m_2)\overline{C^{l+}_{l_1+,\mathrm{i}\rho_2-1/2}(m; m_1', m_2')} = \delta_{m_1 m_1'}\delta_{m_2 m_2'} \tag{2.16}$$

$$\sum_{m=-\infty}^{\infty}\int_0^{\infty}\omega^{-}(\mathrm{i}\rho - \tfrac{1}{2}, \varepsilon)C^{\mathrm{i}\rho-1/2,\varepsilon}_{l_1-,\mathrm{i}\rho_2-1/2}(m; m_1, m_2)\overline{C^{\mathrm{i}\rho-1/2,\varepsilon}_{l_1-,\mathrm{i}\rho_2-1/2}(m; m_1', m_2')}\,\mathrm{d}\rho + \sum_{l=0}^{\infty}\omega_l^{+}\sum_{m=-\infty}^{-l-1}C^{l-}_{l_1-,\mathrm{i}\rho_2-1/2}(m; m_1, m_2)\overline{C^{l-}_{l_1-,\mathrm{i}\rho_2-1/2}(m; m_1', m_2')} = \delta_{m_1 m_1'}\delta_{m_2 m_2'} \tag{2.17}$$

where

$$\omega^{\pm}(\mathrm{i}\rho - \tfrac{1}{2}, \varepsilon) = 2^{-l_1-3/2}\pi^{-1}\rho th\pi\rho \times\{(\mathrm{e}^{\pm(\rho+\rho_2)\pi} - (-1)^{2\varepsilon+l_1})|\Gamma(-l_1 + \mathrm{i}\rho + \mathrm{i}\rho_2)|^2|\gamma(\mathrm{i}\rho_2 - l_1 - \tfrac{1}{2}, \mathrm{i}\rho - \tfrac{1}{2}, \varepsilon)|^2\}^{-1}$$

$$\omega_l^{\pm} = 2^{-l_1-l-9/2}\pi^{-3/2}(2l+1)\Gamma(l+1)\mathrm{e}^{\pm\rho_2\pi}\{|\Gamma(l - l_1 + \mathrm{i}\rho_2 + \tfrac{1}{2})|^2\cosh^2\rho_2\pi\}^{-1}.$$

After a straightforward computation (using the function (I.3.36) as an analytic expression in j_i) we obtain the CG coefficients in the form

$$C^{\mathrm{i}\rho-1/2,\varepsilon}_{l_1+,\mathrm{i}\rho_2-1/2}(m; m_1, m_2) = \delta_{m,m_1+m_2}\mathrm{e}^{\mathrm{i}\pi(m_2-\varepsilon)}2^{(2l_1+2\mathrm{i}\rho_2+3)/4}(2\pi)^{3/2}\gamma(\mathrm{i}\rho_2 - l_1 - \tfrac{1}{2}, \mathrm{i}\rho - \tfrac{1}{2}, \varepsilon) \times\left[\frac{\Gamma(\alpha_{234})\Gamma(\alpha_{012})\Gamma(\alpha_{123})}{\Gamma(\alpha_{134})\Gamma(\alpha_{125})\Gamma(\alpha_{124})}\right]^{1/2}$$

$$\times \frac{\sin\pi(\alpha_{014}/2+\varepsilon)\Gamma(\alpha_{235})\Gamma(\alpha_{013})\Gamma(\alpha_{023})}{\sin\pi\beta_{54}\Gamma(\alpha_{034})\Gamma(\alpha_{234})} F_p(4) \tag{2.18}$$

$$C^{l+}_{l_1+,\mathrm{i}\rho_2-1/2}(m;m_1,m_2) = \delta_{m,m_1+m_2}\mathrm{e}^{-\mathrm{i}\pi\alpha_{014}/2}2^{(2l_1+2\mathrm{i}\rho_2+3)/4}(2\pi)^{3/2} \times \left[\frac{\Gamma(\alpha_{234})\Gamma(\alpha_{012})\Gamma(\alpha_{123})}{\Gamma(\alpha_{134})\Gamma(\alpha_{125})\Gamma(\alpha_{124})}\right]^{1/2}(-1)^{l_1+m+1}\frac{\Gamma(\alpha_{235})\Gamma(\alpha_{013})\Gamma(\alpha_{023})}{\Gamma(\alpha_{034})\Gamma(\alpha_{234})}F_p(4) \tag{2.19}$$

$$C^{\mathrm{i}\rho-1/2,\varepsilon}_{l_1-,\mathrm{i}\rho_2-1/2}(m;m_1,m_2) = \delta_{m,m_1+m_2}\mathrm{e}^{\mathrm{i}\pi(m-\varepsilon)}2^{(2l_1+2\mathrm{i}\rho_2+3)/4}(2\pi)^{3/2}\gamma\left(\mathrm{i}\rho_2-l_1-\tfrac{1}{2},\mathrm{i}\rho-\tfrac{1}{2},\varepsilon\right) \times \left[\frac{\Gamma(\alpha_{025})\Gamma(\alpha_{012})\Gamma(\alpha_{123})}{\Gamma(\alpha_{015})\Gamma(\alpha_{125})\Gamma(\alpha_{124})}\right]^{1/2} \times \frac{\sin\pi(\alpha_{135}/2+\varepsilon)\Gamma(\alpha_{013})\Gamma(\alpha_{024})\Gamma(\alpha_{023})}{\sin\pi\beta_{54}\Gamma(\alpha_{035})\Gamma(\alpha_{025})}F_p(5) \tag{2.20}$$

$$C^{l-}_{l_1-,\mathrm{i}\rho_2-1/2}(m;m_1,m_2) = \delta_{m,m_1+m_2}\mathrm{e}^{-\mathrm{i}\pi\alpha_{245}/2}2^{(2l_1+2\mathrm{i}\rho_2+3)/4}(2\pi)^{3/2} \times \left[\frac{\Gamma(\alpha_{025})\Gamma(\alpha_{012})\Gamma(\alpha_{035})}{\Gamma(\alpha_{015})\Gamma(\alpha_{125})\Gamma(\alpha_{045})}\right]^{1/2}(-1)^{l+m_2}\frac{\Gamma(\alpha_{013})\Gamma(\alpha_{024})\Gamma(\alpha_{023})}{\Gamma(\alpha_{035})\Gamma(\alpha_{025})}F_p(5). \tag{2.21}$$

By using the two-term relations between the Whipple functions [4] one can find large numbers of other expressions for the CG coefficients.

3. Tensor product of complementary series and discrete series representations

We now describe the reduction of the tensor product $T_{\tau_1}\otimes T^{\pm}_{l_2}$ of the complementary series representation T_{τ_1}, $-1<\tau_1<-\frac{1}{2}$, with representation $T^{\pm}_{l_2}$, $l_2=0,1,2,\ldots$ of the discrete series. The approach is analogous to that of section 2; we view $T_{\tau_1}\otimes T^{\pm}_{l_2}$ as the unitarization of a quotient of a tensor product of elementary representations.

Define an operator A on $C^{\infty}(S^1\times S^1)$ by

$$(A_{\tau_1,l_2}f)(s,t) = \frac{2^{\tau_1+l_2}}{\pi}\lim_{j_2\to l_2}\frac{\Gamma(\tau_1+1)\Gamma(j_2+1)}{\Gamma\left(-\frac{2\tau_1+1}{2}\right)\Gamma\left(-\frac{2j_2+1}{2}\right)} \times \int_{S^1\times S^1}|[s_1,s]|^{-\tau_1-1}|[t_1,t]|^{-j_2-1}f(s_1,t_1)\,\mathrm{d}s_1\,\mathrm{d}t_1. \tag{3.1}$$

Let H_{τ_1,l_2} be the Hilbert space completion of $C^{\infty}(S^1\times S^1)/\ker A_{\tau_1,l_2}$ with respect to the scalar product induced by the form

$$(f_1,f_2) = \frac{2^{\tau_1+l_2}}{\pi}\lim_{j_2\to l_2}\frac{\Gamma(\tau_1+1)\Gamma(j_2+1)}{\Gamma\left(-\frac{2\tau_1+1}{2}\right)\Gamma\left(-\frac{2j_2+1}{2}\right)}\int_{S^1\times S^1\times S^1\times S^1}|[s_1,s_2]|^{-\tau_1-1}|[t_1,t_2]|^{-j_2-1} \times f(s_1,t_1)\overline{f(s_2,t_2)}\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}t_1\,\mathrm{d}t_2 \qquad f_1,f_2\in C^{\infty}(S^1\times S^1).$$

Then the tensor product $T_{\tau_1}\otimes(T^+_{l_2}\otimes T^-_{l_2})$ can be realized on the Hilbert space H_{τ_1,l_2}. At $j_1=\tau_1$ and $j_2=l_2$ formula (2.3) gives the representation operator. In H_{τ_1,j_2} there are two invariant subspaces $H^{\pm}_{\tau_1,l_2}$ consisting of elements whose representatives are

$$f(s,t) = \sum_{m_1=-\infty}^{\infty}\sum_{m_2=l_2+1}^{\infty}\alpha_{m_1m_2}\exp(\mathrm{i}m_1\varphi_1+\mathrm{i}m_2\varphi_2) \tag{3.2}$$

and

$$f(s,t)=\sum_{m_1=-\infty}^{\infty}\sum_{m_1=-\infty}^{-l_2-1}\alpha_{m_1m_2}\exp(\mathrm{i}m_1\varphi_1+\mathrm{i}m_2\varphi_2) \tag{3.3}$$

respectively, where $f\in C^{\infty}(S^1\times S^1)$. The subspaces $H^{\pm}_{\tau_1,j_2}$ carry the tensor products $T_{\tau_1}\otimes T^{\pm}_{l_2}$. Now, the following linear combination

$$\tfrac{1}{2}\sum_{\varepsilon=0,1/2}[1+(-1)^{l_2+2\varepsilon}\mathrm{e}^{\pm\mathrm{i}(\tau_1+j)}]\gamma^{-1}(l_2-\tau_1,j,\varepsilon)C^{j\varepsilon}_{\tau_1l_2}f \tag{3.4}$$

is zero on $H^{\pm}_{\tau_1,l_2}$, where $j=\mathrm{i}\rho-\frac{1}{2}$. (Observe, (3.4) at $j=l$, $l=0,1,2,\ldots$ coincides with the expression of $C^{\mp}_{\tau_1l_2}f$ (see equation (I.3.15)).) Consequently, in $T_{\tau_1}\otimes T^{+}_{l_2}[T_{\tau_1}\otimes T^{-}_{l_2}]$ there is exactly one copy of each principal series representation $T_{\mathrm{i}\rho-1/2}$ and there is no negative (positive) discrete series representations $T^{-}_{l}[T^{+}_{l}]$. Thus, the structure of the CG series for $T_{\tau_1}\otimes T^{\pm}_{l_2}$ have the form

$$T_{\tau_1}\otimes T^{\pm}_{l_2}=\int_0^{\infty}T_{\mathrm{i}\rho-1/2}\,\mathrm{d}\rho\oplus\sum_{l=0}^{\infty}T^{\pm}_{l}. \tag{3.5}$$

We have two types of CG coefficients to compute for each case; namely $C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,l_2+}(m;m_1,m_2)$, $C^{l+}_{\tau_1,l_2+}(m;m_1,m_2)$ for $T_{\tau_1}\otimes T^{+}_{l_2}$ and $C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,j_2-}(m;m_1,m_2)$, $C^{l-}_{\tau_1,l_2-}(m;m_1,m_2)$ for $T_{\tau_1}\otimes T^{-}_{l_2}$. One can easily derive the following integral representations for these CG coefficients:

$$C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,l_2\pm}(m;m_1,m_2)=\frac{1}{(2\pi)^{3/2}}\left[\frac{\Gamma(-\tau_1+m_1)\Gamma(l_2\pm m_2)\Gamma(1/2+\mathrm{i}\rho+m)}{\Gamma(1+\tau_1+m_1)\Gamma(1+l_2\pm m_2)\Gamma(1/2-\mathrm{i}\rho+m)}\right]$$
$$\times\int_{S^1\times S^1\times S^1}K_{\varepsilon}(\tau_1s,l_2t;\mathrm{i}\rho-\tfrac{1}{2},u)\exp(\mathrm{i}m_1\varphi_1+\mathrm{i}m_2\varphi_2-\mathrm{i}m\varphi)\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\mathrm{d}\varphi \tag{3.6}$$

$$C^{l\pm}_{\tau_1,l_2\pm}(m;m_1,m_2)=\frac{1}{(2\pi)^{3/2}}\left[\frac{\Gamma(-\tau_1+m_1)\Gamma(-l_2\pm m_2)\Gamma(1+l\pm m)}{\Gamma(1+\tau_1+m_1)\Gamma(1+l_2\pm m_2)\Gamma(-l\pm m)}\right]^{1/2}$$
$$\times\int_{S^1\times S^1\times S^1}K^{\pm}(\tau_1s,l_2t;lu)\exp(\mathrm{i}m_1\varphi_1+\mathrm{i}m_2\varphi_2-\mathrm{i}m\varphi)\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\mathrm{d}\varphi \tag{3.7}$$

where K_{ε} and $K^{\pm}$ are defined by (2.13) and (2.14).

The following completeness relations for CG coefficients hold:

$$\sum_{m=-\infty}^{\infty}\int_0^{\infty}\omega^{+}(\mathrm{i}\rho-\tfrac{1}{2},\varepsilon)C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,l_2+}(m;m_1,m_2)\overline{C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,l_2+}(m;m_1',m_2')}\,\mathrm{d}\rho$$
$$+\sum_{l=0}^{\infty}\omega_l\sum_{m=l+1}^{\infty}C^{l+}_{\tau_1,l_2+}(m;m_1,m_2)\overline{C^{l+}_{\tau_1,l_2+}(m;m_1',m_2')}=\delta_{m_1m_1'}\delta_{m_2m_2'} \tag{3.8}$$

$$\sum_{m=-\infty}^{\infty}\int_0^{\infty}\omega^{-}(\mathrm{i}\rho-\tfrac{1}{2},\varepsilon)C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,l_2-}(m;m_1,m_2)\overline{C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1,l_2-}(m;m_1',m_2')}\,\mathrm{d}\rho$$
$$+\sum_{l=0}^{\infty}\omega_l\sum_{m=-\infty}^{-l-1}C^{l-}_{\tau_1,l_2-}(m;m_1,m_2)\overline{C^{l-}_{\tau_1,l_2-}(m;m_1',m_2')}=\delta_{m_1m_1'}\delta_{m_2m_2'} \tag{3.9}$$

where

$$\omega^{\pm}(\mathrm{i}\rho-\tfrac{1}{2},\varepsilon)=2^{-\tau_1-l_2-3}\pi^{-3}\rho th\pi\rho\frac{\cosh^2\pi\rho\mp\sin^2\pi\tau_1}{\cosh\pi\rho\pm(-1)^{2\varepsilon+l_2}\sin\pi\tau_1}$$

$$\times \left| \frac{\Gamma(\tau_1 + l_2 - \mathrm{i}\rho + 3/2)}{\gamma(l_2 - \tau_1, \mathrm{i}\rho - 1/2, \varepsilon)} \right|^2$$

$$\omega_l = 2^{-\tau_1 - l_2 - l - 5} \pi^{-3/2} \frac{(2l+1)\Gamma(\tau_1 + 1)\Gamma(l_2 + 1)\Gamma(l + \tau_1 + l_2 + 2)}{\sin^2 \pi\tau_1 \Gamma(l - \tau_1 - l_2)\Gamma(l + 1 + \tau_1 - l_2)\Gamma(l + 1 - \tau_1 + l_2)}.$$

The integrals can again be expressed in terms of the ${}_3F_2$ function of unit argument. Hence we obtain for the CG coefficients

$$C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1, l_2+}(m; m_1, m_2) = \delta_{m,m_1+m_2} \mathrm{e}^{\mathrm{i}\pi(m-\varepsilon)} 2^{(\tau_1+l_2+2)/2} (2\pi)^{3/2} \gamma(l_2 - \tau_1, \mathrm{i}\rho - \tfrac{1}{2}, \varepsilon)$$
$$\times \left[\frac{\Gamma(\alpha_{234})\Gamma(\alpha_{012})\Gamma(\alpha_{123})}{\Gamma(\alpha_{134})\Gamma(\alpha_{125})\Gamma(\alpha_{124})} \right]^{1/2}$$
$$\times \frac{\sin\pi(\alpha_{135}/2 + \varepsilon)\Gamma(\alpha_{235})\Gamma(\alpha_{124})\Gamma(\alpha_{125})}{\sin\pi\beta_{41}\Gamma(\alpha_{012})\Gamma(\alpha_{025})} F_p(0) \tag{3.10}$$

$$C^{l+}_{\tau_1, l_2+}(m; m_1, m_2) = \delta_{m,m_1+m_2} \mathrm{e}^{-\mathrm{i}\pi\alpha_{014}/2} 2^{(\tau_1+l_2+2)/2} (2\pi)^{3/2} \left[\frac{\Gamma(\alpha_{234})\Gamma(\alpha_{012})\Gamma(\alpha_{123})}{\Gamma(\alpha_{134})\Gamma(\alpha_{125})\Gamma(\alpha_{124})} \right]^{1/2}$$
$$\times (-1)^{l+m_1} \frac{\Gamma(\alpha_{235})\Gamma(\alpha_{124})\Gamma(\alpha_{125})}{\Gamma(\alpha_{012})\Gamma(\alpha_{025})} F_p(0) \tag{3.11}$$

$$C^{\mathrm{i}\rho-1/2,\varepsilon}_{\tau_1, l_2-}(m; m_1, m_2) = \delta_{m,m_1+m_2} \mathrm{e}^{\mathrm{i}\pi(m_1-\varepsilon)} 2^{(\tau_1+l_2+2)/2} (2\pi)^{3/2} \gamma(l_2 - \tau_1, \mathrm{i}\rho - \tfrac{1}{2}, \varepsilon)$$
$$\times \left[\frac{\Gamma(\alpha_{234})\Gamma(\alpha_{034})\Gamma(\alpha_{123})}{\Gamma(\alpha_{134})\Gamma(\alpha_{345})\Gamma(\alpha_{124})} \right]^{1/2}$$
$$\times \frac{\sin\pi(\alpha_{245}/2 + \varepsilon)\Gamma(\alpha_{235})\Gamma(\alpha_{134})\Gamma(\alpha_{345})}{\sin\pi\beta_{14}\Gamma(\alpha_{034})\Gamma(\alpha_{035})} F_p(0) \tag{3.12}$$

$$C^{l-}_{\tau_1, l_2-}(m; m_1, m_2) = \delta_{m,m_1+m_2} \mathrm{e}^{-\mathrm{i}\pi\alpha_{245}/2} 2^{(\tau_1+l_2+2)/2} (2\pi)^{3/2} \left[\frac{\Gamma(\alpha_{234})\Gamma(\alpha_{034})\Gamma(\alpha_{035})}{\Gamma(\alpha_{134})\Gamma(\alpha_{345})\Gamma(\alpha_{045})} \right]^{1/2}$$
$$\times (-1)^{l_2+m} \frac{\Gamma(\alpha_{235})\Gamma(\alpha_{134})\Gamma(\alpha_{345})}{\Gamma(\alpha_{034})\Gamma(\alpha_{035})} F_p(0). \tag{3.13}$$

In deriving these equations we have used the following two-term relations:

$$F_p(1) = \frac{\Gamma(\alpha_{124})\Gamma(\alpha_{125})\Gamma(\alpha_{123})}{\Gamma(\alpha_{024})\Gamma(\alpha_{025})\Gamma(\alpha_{023})} F_p(0) \qquad \text{when} \quad \alpha_{345} = 1 + l_2 - m_2 \leqslant 0 \quad \text{integer}$$

and

$$F_p(4) = \frac{\Gamma(\alpha_{134})\Gamma(\alpha_{234})\Gamma(\alpha_{345})}{\Gamma(\alpha_{013})\Gamma(\alpha_{023})\Gamma(\alpha_{035})} F_p(0) \qquad \text{when} \quad \alpha_{125} = 1 + l_2 + m_2 \leqslant 0 \quad \text{integer}.$$

4. Tensor products of discrete series representations

We start with the tensor product $V_{l_1,l_2} = (T^+_{l_1} \oplus T^-_{l_1}) \otimes (T^+_{l_2} \oplus T^-_{l_2})$. This tensor product is realized on the Hilbert space H_{l_1,l_2} obtained by completing of $C^\infty(S^1 \times S^1)/\ker A_{l_1,l_2}$ with respect to the scalar product induced by the form

$$(f_1, f_2) = \frac{2^{l_1+l_2}}{\pi} \lim_{j_1 \to l_1, j_2 \to l_2} \frac{\Gamma(j_1+1)\Gamma(j_2+1)}{\Gamma\left(-\frac{2j_1+1}{2}\right)\Gamma\left(-\frac{2j_2+1}{2}\right)} \int_{S^1\times S^1\times S^1\times S^1} |[s_1, s_2]|^{-j_1-1} |[t_1, t_2]|^{-j_2-1}$$
$$\times f(s_1, t_1)\overline{f(s_2, t_2)}\, \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}t_1\, \mathrm{d}t_2 \qquad f_1, f_2 \in C^\infty(S \times S)$$

where an operator A_{l_1,l_2} is defined by

$$(A_{l_1,l_2}f)(s,t) = \frac{2^{l_1+l_2}}{\pi} \lim_{j_1\to l_1, j_2\to l_2} \frac{\Gamma(j_1+1)\Gamma(j_2+1)}{\Gamma\left(-\frac{2j_1+1}{2}\right)\Gamma\left(-\frac{2j_2+1}{2}\right)} \times \int_{S^1\times S^1} |[s_1,s]|^{-j_1-1}|[t_1,t]|^{-j_2-1} f(s_1,t_1)\,\mathrm{d}s_1\,\mathrm{d}t_1.$$

The G-action on H_{l_1,l_2} is given by (2.3), where now $j_1 = l_1$ and $j_2 = l_2$. Let $H^{++}_{l_1,l_2}$, $H^{--}_{l_1,l_2}$, $H^{+-}_{l_1,l_2}$ and $H^{-+}_{l_1,l_2}$ be subspaces of H_{l_1,l_2} consisting of elements with representative $f \in C^\infty(S^1\times S^1)$ whose Fourier coefficients $\alpha_{m_1m_2}$ in the Fourier series

$$f(s,t) = \sum_{m_1m_2}^{\infty} \alpha_{m_1m_2} \exp(\mathrm{i}m_1\varphi_1 + \mathrm{i}m_2\varphi_2) \tag{4.1}$$

are zero for $\{m_1,m_2 \in Z\colon m_1 \leqslant l_1, m_2 \leqslant l_2\}$, $\{m_1,m_2 \in Z\colon m_1 \geqslant -l_1, m_2 \geqslant -l_2\}$, $\{m_1,m_2 \in Z\colon m_1 \leqslant l_1, m_2 \geqslant -l_2\}$ and $\{m_1,m_2 \in Z\colon m_1 \geqslant -l_1, m_2 \leqslant l_2\}$ respectively. The subspaces $H^{++}_{l_1,l_2}$, $H^{--}_{l_1,l_2}$, $H^{+-}_{l_1,l_2}$ and $H^{-+}_{l_1,l_2}$ are invariant under the action of the representation V_{l_1,l_2}. The tensor products $T^\pm_{l_1}\otimes T^\pm_{l_2}$ and $T^\pm_{l_1}\otimes T^\mp_{l_2}$ are obtained by restriction V_{l_1,l_2} to the subspaces $H^{\pm\pm}_{l_1,l_2}$ and $H^{\pm\mp}_{l_1,l_2}$, respectively. It can be shown [2, 3] that the structure of CG series have the form

$$T^\pm_{l_1}\otimes T^\pm_{l_2} = \sum_{l=l_1+l_2+1}^{\infty} \oplus T^\pm_l \tag{4.2}$$

$$T^\pm_{l_1}\otimes T^\mp_{l_2} = \int_0^\infty T_{\mathrm{i}\rho_2-1/2}\,\mathrm{d}\rho \oplus \sum_{l=0}^{l_2-l_1-1} T^\mp_l. \tag{4.3}$$

In the last equation it is understood that $l_2 \geqslant l_1$ and there are no discrete series $T^\mp_l$ if $l_2 = l_1$.

First we dispose of the case of two discrete series representations, both with '+' sign or both with '−' sign. In this case the following relation holds:

$$C^{l\pm}_{l_1\pm,l_2\pm}(T^\pm_{l_1}\otimes T^\pm_{l_2})(g) = T^\pm_l(g)C^{l\pm}_{l_1\pm,l_2\pm} \tag{4.4}$$

where the intertwining operators $C^{l+}_{l_1+,l_2+}$ and $C^{l-}_{l_1-,l_2-}$ are defined by

$$(C^{l\pm}_{l_1\pm,l_2\pm}f)(u) = \int \tilde{K}^\pm(l_1s,l_2t;lu)f(s,t)\,\mathrm{d}s\,\mathrm{d}t \tag{4.5}$$

where

$$\tilde{K}^\pm(l_1s,l_2t;lu) = \left.\frac{\partial}{\partial z}\right|_{z=l_2-l_1} K^\pm(j_1s,j_2t;lu) \tag{4.6}$$

for $K^\pm$, see (2.14). This then gives the integral representations for CG coefficients of the tensor products $T^\pm_{l_1}\otimes T^\pm_{l_2}$. As a result:

$$C^{l+}_{l_1+,l_2+}(m;m_1,m_2) = 1/(2\pi)^{3/2}\left[\frac{\Gamma(-l_1+m_1)\Gamma(-l_2+m_2)\Gamma(1+l+m)}{\Gamma(1+l_1+m_1)\Gamma(1+l_2+m_2)\Gamma(-l+m)}\right]^{1/2} \times \int_{S^1\times S^1\times S^1} \tilde{K}^+(l_1s,l_2t;lu)f(s,t)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u \tag{4.7}$$

$$C^{l-}_{l_1-,l_2-}(m;m_1,m_2) = 1/(2\pi)^{3/2}\left[\frac{\Gamma(-l_1-m_1)\Gamma(-l_2-m_2)\Gamma(1+l-m)}{\Gamma(1+l_1-m_1)\Gamma(1+l_2-m_2)\Gamma(-l-m)}\right]^{1/2} \times \int_{S^1\times S^1\times S^1} \tilde{K}^-(l_1s,l_2t;lu)f(s,t)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u. \tag{4.8}$$

Moreover, from equation (9.8) of [2], one can obtain the following completeness relations for CG coefficients:

$$\sum_{l=l_1+l_2+1}^{\infty}\sum_{m=l+1}^{\infty}\omega_l C^{l+}_{l_1+,l_2+}(m;m_1,m_2)\overline{C^{l+}_{l_1+,l_2+}(m;m_1',m_2')}=\delta_{m_1m_1'}\delta_{m_2m_2'} \tag{4.9}$$

$$\sum_{l=l_1+l_2+1}^{\infty}\sum_{m=-\infty}^{-l-1}\omega_l C^{l-}_{l_1-,l_2-}(m;m_1,m_2)\overline{C^{l-}_{l_1-,l_2-}(m;m_1',m_2')}=\delta_{m_1m_1'}\delta_{m_2m_2'} \tag{4.10}$$

where

$$\omega_l=2^{-l_1-l_2-l-5}(2l+1)\frac{l!(l+l_1+l_2+1)!}{(l-l_1-l_2-1)!(l+l_1-l_2)!(l-l_1+l_2)!}.$$

Thus, we can calculate the CG coefficients in a straightforward manner from its integral representation.

First we calculate the CG coefficients for tensor product $T^+_{l_1}\otimes T^+_{l_2}$. It is easy to see that, the integral appearing in (4.7) is obtained by differentiating the integral in (3.24.I), with respect to z and then putting $z=l_2-l_1$. Hence, we can use the expression (3.39.I) to calculate the CG coefficients under consideration. Instead of using equation (3.39.I) directly, it is convenient to change it using the relation (see, equation (B.10) of I)

$$F_p(1)=\frac{\Gamma(\alpha_{012})\Gamma(\alpha_{125})\Gamma(\alpha_{123})}{\Gamma(\alpha_{024})\Gamma(\alpha_{245})\Gamma(\alpha_{234})}F_p(4)\qquad \text{when}\quad \alpha_{035}=1+l-m\leqslant 0\quad \text{integer}. \tag{4.11}$$

It now follows from (3.39.I) that the integrals on the right-hand side of (3.24.I) have the value

$$\delta_{m,m_1+m_2}\mathrm{e}^{\mathrm{i}\pi(m_1+\alpha_{245}/2)}2^{\frac{2+j_1+j_2}{2}}(2\pi)^3\frac{\Gamma(\alpha_{235})\Gamma(\alpha_{023})\Gamma(\alpha_{125})}{\Gamma(\alpha_{245})\Gamma(\alpha_{234})}F_p(4) \tag{4.12}$$

which implies (in table 1 the z dependencies are made explicit)

$$C^{l+}_{l_1+l_2+}(m;m_1,m_2)=\delta_{m,m_1+m_2}\mathrm{e}^{\mathrm{i}\pi(m_1-\alpha_{245}/2)}2^{\frac{2+l_1+l_2}{2}}(2\pi)^{3/2}\left[\frac{\Gamma(\alpha_{234})\Gamma(\alpha_{012})\Gamma(\alpha_{123})}{\Gamma(\alpha_{134})\Gamma(\alpha_{125})\Gamma(\alpha_{124})}\right]^{1/2}\times\frac{\Gamma(\alpha_{235})\Gamma(\alpha_{023})\Gamma(\alpha_{125})\Gamma(\alpha_{013})}{\Gamma(\alpha_{234})}F_p(4). \tag{4.13}$$

In arriving at this result we have used the facts

$$\left.\frac{1}{\Gamma(\alpha_{245})}\right|_{z=l_2-l_1}=0 \tag{4.14}$$

and

$$\left.\frac{\partial}{\partial z}\frac{1}{\Gamma(\alpha_{245})}\right|_{z=l_2-l_1}=(-1)^{\alpha_{013}-1}\Gamma(\alpha_{013}) \tag{4.15}$$

provided that $l=l_1+l_2+1, l_1+l_2+2,\ldots$. The last equality is essentially the consequence of

$$\lim_{z\to-n}\frac{\psi(z)}{\Gamma(z)}=(-1)^{n+1}n!\qquad n=0,1,2,\ldots \tag{4.16}$$

which is easily proved by making use of equations (1.17.11) and (1.17.12) from [5]. Here $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function.

Table 1. Showing α, β.

$\alpha_{012} = m_2 - (j_1 + j_2 + z)/2$	$\alpha_{034} = -m_2 - (j_1 + j_2 + z)/2$	$\alpha_{135} = 2 + j_1 + j_2 + j$
$\alpha_{013} = 1 - z + j$	$\alpha_{035} = 1 + j - m$	$\alpha_{145} = 1 + j_1 + j_2 - j$
$\alpha_{014} = -z - j$	$\alpha_{045} = -j - m$	$\alpha_{234} = m_1 + (j_1 + j_2 - z)/2$
$\alpha_{015} = 1 - m_1 + (j_1 + j_2 - z)/2$	$\alpha_{123} = 1 + j + m$	$\alpha_{235} = 1 + z + j$
$\alpha_{023} = -j_1 - j_2 + j$	$\alpha_{124} = -j + m$	$\alpha_{245} = z - j$
$\alpha_{024} = -1 - j_1 - j_2 - j$	$\alpha_{125} = 1 + m_2 + (j_1 + j_2 + z)/2$	$\alpha_{345} = 1 - m_1 + (j_1 + j_2 + z)/2$
$\alpha_{025} = -m_1 - (j_1 + j_2 - z)/2$	$\alpha_{134} = 1 + m_1 + (j_1 + j_2 - z)/2$	
$\beta_{01} = -j_1 - j_2 - m$	$\beta_{15} = 1 - z + m$	
$\beta_{02} = 1 - z - m$	$\beta_{23} = -j + m + (j_1 + j_2 - z)/2$	
$\beta_{03} = -j - m_1 - (j_1 + j_2 + z)/2$	$\beta_{24} = 1 + j + m_2 - (j_1 + j_2 - z)/2$	
$\beta_{04} = 1 - j - m_1 - (j_1 + j_2 + z)/2$	$\beta_{25} = -j_1 - j_2 + m$	
$\beta_{05} = -j_1 - j_2 - z$	$\beta_{34} = 2 + 2j$	
$\beta_{12} = 2 + j_1 + j_2 - z$	$\beta_{35} = 1 + j + m_1 - (j_1 + j_2 + z)/2$	
$\beta_{13} = 1 - j + m_2 + (j_1 + j_2 - z)/2$	$\beta_{45} = -j + m_1 - (j_1 + j_2 + z)/2$	
$\beta_{14} = 2 + j + m_2 + (j_1 + j_2 - z)/2$		

In a similar fashion the CG coefficients for the case $D_{\tilde{l}_1} \otimes D_{\tilde{l}_2}$ turn

$$C^{l-}_{l_1-l_2-}(m; m_1, m_2) = \delta_{m,m_1+m_2}(2\pi)^{3/2} e^{i\pi(m_1+1-\alpha_{014}/2)} 2^{\frac{2+l_1+l_2}{2}} \left[\frac{\Gamma(\alpha_{025})\Gamma(\alpha_{034})\Gamma(\alpha_{035})}{\Gamma(\alpha_{015})\Gamma(\alpha_{345})\Gamma(\alpha_{045})}\right]^{1/2} \times \frac{\Gamma(\alpha_{013})\Gamma(\alpha_{023})\Gamma(\alpha_{015})\Gamma(\alpha_{235})}{\Gamma(\alpha_{034})} F_p(4). \qquad (4.17)$$

Turning now to the final case $T^{\pm}_{l_1} \otimes T^{\mp}_{l_2}$, there are four types of CG coefficients to be calculated; namely $C^{i\rho-1/2,\varepsilon}_{l_1+,l_2-}(m; m_1, m_2)$, $C^{l-}_{l_1+,l_2-}(m; m_1, m_2)$ for $T^{+}_{l_1} \otimes T^{-}_{l_2}$ and $C^{i\rho-1/2}_{l_1-,l_2+}(m; m_1, m_2)$, $C^{l+}_{l_1-,l_2+}(m; m_1, m_2)$ for $T^{-}_{l_1} \otimes T^{+}_{l_2}$, where ε enumerates two (linearly dependent) CG coefficients corresponding to $T^{\pm}_{l_1} \otimes T^{\mp}_{l_2} \to T_{i\rho-1/2}$ (recall that, each UIRs $T_{i\rho-1/2}$ occurs once in $T^{\pm}_{l_1} \otimes T^{\mp}_{l_2}$). The results of section 9 of [2] lead immediately to the following integral representations for these CG coefficients:

$$C^{i\rho-1/2,\varepsilon}_{l_1\pm,l_2\mp}(m; m_1, m_2) = \frac{1}{(2\pi)^{3/2}} \left[\frac{\Gamma(-l_1 \pm m_1)\Gamma(-l_2 \mp m_2)\Gamma(1/2 + i\rho + m)}{\Gamma(1 + l_1 \pm m_1)\Gamma(1 + l_2 \mp m_2)\Gamma(1/2 - i\rho + m)}\right]^{1/2} \times \int_{S^1\times S^1\times S^1} K_\varepsilon(l_1 s, l_2 t; i\rho - \tfrac{1}{2}, u) \exp(im_1\varphi_1 + im_2\varphi_2 - im\varphi)\, d\varphi_1\, d\varphi_2\, d\varphi \qquad (4.18)$$

$$C^{l\mp}_{l_1\pm,l_2\mp}(m; m_1, m_2) = \frac{1}{(2\pi)^{3/2}} \left[\frac{\Gamma(-l_1 \pm m_1)\Gamma(-l_2 \mp m_2)\Gamma(1 + l \mp m)}{\Gamma(1 + l_1 \pm m_1)\Gamma(1 + l_2 \mp m_2)\Gamma(l \mp m)}\right]^{1/2} \times \int_{S^1\times S^1\times S^1} K^{\mp}(l_1 s, l_2 t; lu) \exp(im_1\varphi_1 + im_2\varphi_2 - im\varphi)\, d\varphi_1\, d\varphi_2\, d\varphi. \qquad (4.19)$$

Hence, we obtain for the CG coefficients

$$C^{i\rho-1/2,\varepsilon}_{l_1+,l_2-}(m; m_1, m_2) = \delta_{m,m_1+m_2} e^{i\pi(m_1-\varepsilon)} 2^{\frac{2+l_1+l_2}{2}} (2\pi)^{3/2} \gamma(l_2 - l_1, i\rho - \tfrac{1}{2}, \varepsilon) \times \left[\frac{\Gamma(\alpha_{234})\Gamma(\alpha_{034})\Gamma(\alpha_{123})}{\Gamma(\alpha_{134})\Gamma(\alpha_{345})\Gamma(\alpha_{124})}\right]^{1/2} \frac{\sin\pi(\alpha_{245} + \varepsilon)\Gamma(\alpha_{235})\Gamma(\alpha_{134})\Gamma(\alpha_{345})}{\sin\pi\beta_{14}\Gamma(\alpha_{034})\Gamma(\alpha_{035})} F_p(0) \qquad (4.20)$$

$$C^{i\rho-1/2}_{l_1-,l_2+}(m; m_1, m_2) = \delta_{m,m_1+m_2} e^{i\pi(m-\varepsilon)} 2^{\frac{2+l_1+l_2}{2}} (2\pi)^{3/2} \gamma(l_2 - l_1, i\rho - \tfrac{1}{2}, \varepsilon)$$

$$\times\left[\frac{\Gamma(\alpha_{025})\Gamma(\alpha_{012})\Gamma(\alpha_{123})}{\Gamma(\alpha_{015})\Gamma(\alpha_{125})\Gamma(\alpha_{124})}\right]^{1/2}$$
$$\times\frac{\sin\pi(\alpha_{135}/2+\varepsilon)\Gamma(\alpha_{235})\Gamma(\alpha_{124})\Gamma(\alpha_{125})}{\sin\pi\beta_{41}\Gamma(\alpha_{012})\Gamma(\alpha_{025})}F_p(0) \tag{4.21}$$

$$C^{l-}_{l_1+,l_2-}(m;m_1,m_2)=\delta_{m,m_1+m_2}\mathrm{e}^{-\mathrm{i}\pi\alpha_{245}/2}2^{\frac{2+l_1+l_2}{2}}(2\pi)^{3/2}\left[\frac{\Gamma(\alpha_{234})\Gamma(\alpha_{034})\Gamma(\alpha_{035})}{\Gamma(\alpha_{134})\Gamma(\alpha_{345})\Gamma(\alpha_{045})}\right]^{1/2}$$
$$\times(-1)^{l_2+m}\frac{\Gamma(\alpha_{235})\Gamma(\alpha_{134})\Gamma(\alpha_{345})}{\Gamma(\alpha_{034})\Gamma(\alpha_{035})}F_p(0) \tag{4.22}$$

and

$$C^{l+}_{l_1-,l_2+}(m;m_1,m_2)=\delta_{m,m_1+m_2}\mathrm{e}^{-\mathrm{i}\pi\alpha_{014}/2}2^{\frac{2+l_1+l_2}{2}}(2\pi)^{3/2}\left[\frac{\Gamma(\alpha_{025})\Gamma(\alpha_{012})\Gamma(\alpha_{123})}{\Gamma(\alpha_{015})\Gamma(\alpha_{125})\Gamma(\alpha_{124})}\right]^{1/2}$$
$$\times(-1)^{l+m_1}\frac{\Gamma(\alpha_{235})\Gamma(\alpha_{124})\Gamma(\alpha_{125})}{\Gamma(\alpha_{012})\Gamma(\alpha_{025})}F_p(0). \tag{4.23}$$

We conclude this section by writing the completeness relations for these CG coefficients:

$$\sum_{m=-\infty}^{\infty}\int_0^{\infty}\omega(\mathrm{i}\rho-\tfrac{1}{2},\varepsilon)C^{\mathrm{i}\rho-1/2,\varepsilon}_{l_1+,l_2-}(m;m_1,m_2)\overline{C^{\mathrm{i}\rho-1/2,\varepsilon}_{l_1+,l_2-}(m;m_1',m_2')}\,\mathrm{d}\rho$$
$$+\sum_{l=0}^{l_2-l_1-1}\omega_l\sum_{m=-\infty}^{-l-1}C^{l-}_{l_1+,l_2-}(m;m_1,m_2)\overline{C^{l-}_{l_1+,l_2-}(m;m_1',m_2')}$$
$$=\delta_{m_1m_1'}\delta_{m_2m_2'} \tag{4.24}$$

$$\sum_{m=-\infty}^{\infty}\int_0^{\infty}\omega(\mathrm{i}\rho-\tfrac{1}{2},\varepsilon)C^{\mathrm{i}\rho-1/2,\varepsilon}_{l_1-,l_2+}(m;m_1,m_2)\overline{C^{\mathrm{i}\rho-1/2,\varepsilon}_{l_1-,l_2+}(m;m_1',m_2')}\,\mathrm{d}\rho$$
$$+\sum_{l=0}^{l_2-l_1-1}\omega_l\sum_{m=l+1}^{\infty}C^{l+}_{l_1-,l_2+}(m;m_1,m_2)\overline{C^{l+}_{l_1-,l_2+}(m;m_1',m_2')}$$
$$=\delta_{m_1m_1'}\delta_{m_2m_2'} \tag{4.25}$$

where

$$\omega(\mathrm{i}\rho-\tfrac{1}{2},\varepsilon)=2^{-l_1-l_2-5}\pi^{-3}(l_1!l_2!)^{-1}\sinh\pi\rho|\Gamma(l_1+l_2+\mathrm{i}\rho+\tfrac{3}{2})|^2\gamma(l_2-l_1,\mathrm{i}\rho-\tfrac{1}{2},\varepsilon)$$
$$\omega_l=2^{-l_1-l_2-l-5}\pi^{-7/2}(2l+1)\frac{l!(l_1+l_2-l)!(l_2-l_1-l-1)!(l+l_1+l_2+1)!}{(l-l_1+l_2)!}.$$

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